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COMMON FIXED POINT THEOREMS FOR THREE SELFMAPS OF A COMPLETE D*- METRIC SPACE

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ABSTRACT

Suppose (X, D) is a D*- metric space and P, Q and T are selfmaps of X. If these three maps and the space X satisfy certain conditions, we shall prove that they have a unique common fixed point in this paper. As a consequence we deduce a common fixed point theorem for three selfmaps of a complete D*- metric space. Further, we show that a common fixed point theorem for three selfmaps of a metric space proved by S. L. Singh and S. P. Singh ([9]) follows as a particular case of the theorem.*

Mathematics Subject Classification: *47H10, 54H25.*

Key Words: *D*-metric space; Compatible; Fixed point theorem*

1. INTRODUCTION AND PRELIMINARIES

The study of contractive type conditions through metric spaces in fixed point theory plays vital role because it finds many applications in different areas like differential equations, integral equations, game theory, operational research and mathematical economics.

Different mathematicians tried to generalize the usual notion of metric space (X, d). In 1992 Dhage [2] has initiated the study of generalized metric space called D- metric space and fixed point theorems for selfmaps of such spaces. Later researchers have made a significant contribution to fixed point of D- metric spaces in [1], [3], and [4]. Unfortunately almost all the fixed point theorems proved on D-metric spaces are not valid in view of papers [5], [6] and [7].

Recently Shaban Sedghi, Nabi Shobe and Haiyun Zhou [8], have introduced D*- metric spaces as a probable modification of D- metric spaces and proved some fixed point theorems.

Definition 1.1 ([8]): Let X be a non-empty set. A function $D^*: X^3 \rightarrow [0, \infty)$ is said to be a **generalized metric** or **D*-metric** or **G-metric** on X, if it satisfies the following conditions

(i) $D^*(x, y, z) \geq 0$ for all $x, y, z \in X$.

(ii) $D^*(x, y, z) = 0$ if and only if $x = y = z$.

(iii) $D^*(x, y, z) = D^*(\sigma(x, y, z))$ for all $x, y, z \in X$

where $\sigma(x, y, z)$ is any permutation of the set $\{x, y, z\}$.

(iv) $D^*(x, y, z) \leq D^*(x, y, w) + D^*(w, z, z)$ for all $x, y, z, w \in X$.

The pair (X, D *), where D * is a generalized metric on X is called a **D*–metric space** or a **generalized metric space**.

Example 1. 2: Let (X, d) be a metric space. Define $D_1^*: X^3 \to [0, \infty)$ by

$D_1^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ for x, y, z ϵ X. Then (X, D_1^*) is a generalized metric space.

Example 1.3: Let (X, d) be a metric space. Define $D_2^*: X^3 \to [0, \infty)$ by

$D_2^*(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for x, y, z ϵ X. Then (X, D_2^*) is a generalized metric space.

Example 1.4: Let X = R, define $D^*: R^3 \to [\mathbf{0}, \infty)$ by

$$D^*(x, y, z) = \begin{cases} 0 & \text{if } x = y = z \\ \max\{x, y, z\} & \text{otherwise} \end{cases}$$

Then (R, D *) is a generalized metric space.

Note 1.5: Using the inequality in (iv) and (ii) of Definition 1.1, one can prove that if (X, D *) is a D*–metric space, then $D^*(x, x, y) = D^*(x, y, y)$ for all $x, y, \in X$.

Infact $D^*(x, x, y) \leq D^*(x, x, x) + D^*(x, y, y) = D^*(x, y, y)$ and

$D^*(y, y, x) \leq D^*(y, y, y) + D^*(y, x, x) = D^*(y, x, x)$, proving the inequity.

Definition 1.6: Let (X, D*) be a D*-metric space. For x ϵ X and r > 0, the set $B_{D^*}(x, r) = \{y \in X; D^*(x, y, y) < r\}$ is called the **open ball** of radius r about x.

For example, if X = R and $D^*: R^3 \to [0, \infty)$ is defined by

$D^*(x, y, z) = |x - y| + |y - z| + |z - x|$ for all x, y, z ϵ R. Then

$B_{D^*}(0, 1) = \{y \in R; D^*(0, y, y) < 1\}$

$= \{y \in R; 2|y| < 1\}$

$= \{y \in R; |y| < ½\} = (-½, ½)$.

Definition 1.7: Let (X, D *) be a D*–metric space and E ⊂ X.

(i) If for every x ∈ E, there is a δ > 0 such that $B_{D^*}(x, \delta) \subset E$, then E is said to be an **open subset** of X

(ii) If there is a k > 0 such that $D^*(x, y, y) < k$ for all x, y ∈ E then E is said to be **D*–bounded.** It has been observed in [9] that, if τ is the set of all open sets in (X, D*), then τ is a topology on X (called the **topology induced by the D*–metric**) and also proved that B_{D*} (x, r) is an open set for each x ∈ X and r > 0 ([8], Lemma 1.5). If (X, τ) is a compact topological space we shall call (X, D *) is a **compact** D*–metric space.

Definition 1.8: Let (X, D *) be a D*–metric space. A sequence $\{x_n\}$ in X is said to

(i) **converge to x** if $D^*(x_n, x_n, x) = D^*(x_n, x, x) \to 0$ as $n \to \infty$

(ii) be a **Cauchy sequence**, if to each ∈ > 0, there is a natural number n_0 such that $D^*(x_n, x_n, x_m) < \in$ for all $m, n \geq n_0$.

It is easy to see (infact proved in [8], Lemma 1.8 and Lemma 1.9) that, if $\{x_n\}$ converges to x in(X, D*) then x is unique and that $\{x_n\}$ is a Cauchy sequence in (X, D*). However, a Cauchy sequence in a (X, D *) need not be convergent as shown in the example given below.

Example 1.9: Let X = (0, 1] and $D^*(x, y, z) = | x - y | + | y - z | + | z - x |$ for x, y, z ∈ X, so that (X, D *) is a D*–metric space.

Define $x_n = \frac{1}{n}$ for n = 1, 2, 3 ……., then $D^*(x_n, x_n, x_m) = 2 | x_n - x_m | = 2 | \frac{1}{n} - \frac{1}{m} |$, so that

$D^*(x_n, x_n, x_m) \to 0$ as $m, n \to \infty$, proving $\{x_n\}$ is a Cauchy sequence in (X, D*). Clearly $\{x_n\}$ does not converge to any point in X.

Definition 1.10: A D*–metric space (X, D *) is said to **complete** if every Cauchy sequence in it converges to some point in it.

It follows that the D*–metric space given in Example 1.9 is not complete.

Note 1.11: We have seen (In Example 1.2 and Example 1.3) that on any metric space (X, d) it is possible to define at least two D*–metrics, namely D_1* and D_2 *, using the metric d. We shall call D_1 * and D_2 * as D*–metrics induced by d. Thus every metric space (X, d) gives rise to at least two D*–metric spaces (X, D_1 *) and (X, D_2*). Also if (X, D *) is a D*–metric then defining $d_0(x, y) = D^*(x, y, y)$ for x, y ∈ X, we can show easily that (X, d_0) is a metric space and we shall call d_0 as a metric induced by D *.

The following result is of use for our discussion.

Theorem 1.12: Let (X, d) be a metric space and D_i *(i =1, 2) be the two D*– metrics induced by d (given in Example 1.2 and Example 1.3). For any i (=1, 2) a sequence $\{x_n\}$ in (X, D_i *) is a Cauchy sequence if and only if $\{x_n\}$ is a Cauchy sequence in (X, d).

Proof: - First note that for i =1, 2 we have

$d(x, y) \leq D_i^*(x, y, y) \leq 2d(x, y)$ for all x, y ∈ X .

Now the theorem follows immediately in view of the above inequality.

For example, if $\{x_n\}$ is a Cauchy sequence in (X , d), then for any given $\in > 0$ choose a natural number n_0 such that $m , n \geq n_0$ implies $d (x_m, x_n) < \frac{\in}{2}$; and note that for the same n_0 we have $m, n \geq n_0$ implies $D_i{}^*(x_m , x_n , x_n) \leq 2d (x_m\, x_n) < \in$,

proving that $\{x_n \}$ is a Cauchy sequence in $(X , D_i{}^*)$.

Similarly the other part of the theorem can be proved using the other inequality noted in the beginning of the proof.

Corollary1.13: Suppose (X, d) is a metric space. Let $D_1{}^*$ and $D_2{}^*$ be two D*– metrics induced by d, then for any i (=1, 2) the space $(X, D_i{}^*)$ is complete if and only if (X, d) is complete.

Proof: - Follows from Theorem 1.12.

Definition 1.14: If (X, D*) is a D*-metric space, then D* is a **continuous function** on X^3, in the sense that $\lim_{n\to\infty} D^*(x_n, y_n, z_n) = D^*(x, y, z)$, whenever $\{(x_n, y_n, z_n)\}$ in X^3 converges to $(x, y, z) \in X^3$. Equivalently,

$$\lim_{n\to\infty} x_n = x,\ \lim_{n\to\infty} y_n = y,\ \lim_{n\to\infty} z_n = z \Leftrightarrow \lim_{n\to\infty} D^*(x_n,y_n,z_n) = D^*(x,y,z).$$

Notation: For any selfmap T of X, we denote T(x) by Tx.

If S and T are selfmaps of a set X, then any $z \in X$ such that Sz = Tz = z is called a **common fixed point** of S and T.

Two selfmaps S and T of X are said to be **commutative** if ST = TS where ST is their composition SoT defined by (SoT) x = STx for all $x \in X$.

Definition 1.15: Suppose S and T are selfmaps of a D*–metric space (X, D*) satisfying the condition $T(X) \subseteq S(X)$. Then for any $x_0 \in X$, $Tx_0 \in T(X)$ and hence $Tx_0 \in S(X)$, so that there is a $x_1 \in X$ with $Tx_0 = Sx_1$, since $T(X) \subseteq S(X)$. Now $Tx_1 \in T(X)$ and hence there is a $x_2 \in X$ with $Tx_2 \in T(X) \subseteq S(X)$ so that $Tx_1 = Sx_2$. Again $Tx_2 \in T(X)$ and hence $Tx_2 \in S(X)$ with $Tx_2 = Sx_3$. Thus repeating this process to each $x_0 \in X$, we get a sequence $\{x_n\}$ in X such that $Tx_n = Sx_{n+1}$ for $n \geq 0$. We shall call this sequence as an **associated sequence of x_0 relative to the two selfmaps S and T.** It may be noted that there may be more than one associated sequence for a point $x_0 \in X$ relative to selfmaps S and T.

Let S and T are selfmaps of a D*-metric space (X, D*) such that $T(X) \subseteq S(X)$. For any $x_o \in X$, if $\{x_n\}$ is a sequence in X such that $Tx_n = Sx_{n+1}$ for $n \geq 0$, then $\{x_n\}$ is called an **associated sequence** of x_0 relative to the two selfmaps S and T.

Definition 1.16: A function $\emptyset: [0, \infty) \to [0, \infty)$ is said to be a **contractive modulus**, if $\emptyset(0) = 0$ and $\emptyset(t) < t$ for $t > 0$.

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Definition 1.17: A real valued function Ø defined on $X \subseteq R$ is said to be **upper semi continuous**, if $\lim_{n\to\infty} \sup Ø(t_n) \leq Ø(t)$ for every sequence $\{t_n\}$ in X with $t_n \to t$ as $n \to \infty$.

Definition 1.18: If S and T are selfmaps of a D*-metric space (X, D*) such that for every sequence $\{x_n\}$ in X with $\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t$, we have

$\lim_{n\to\infty} D^*(STx_n, TSx_n, TSx_n) = 0$, then we say that S and T are **compatible**

2. THE MAIN RESULTS:

2. 1 Theorem: Let P, Q and T are selfmaps of a D*- metric space (X, D*) satisfying the conditions

(i) $P(X) \cup Q(X) \subseteq T(X)$

(ii) $D^*(Px, Qy, Qy) \leq Ø(\lambda(x, y))$ for all $x, y \in X$

where Ø is an upper semi continuous and contractive modulus

and

(ii)′ $\lambda(x, y) = \max\{D^*(Tx, Ty, Ty), D^*(Px, Tx, Tx), D^*(Qy, Ty, Ty),$

$$\frac{1}{2}[D^*(Px, Ty, Ty) + D^*(Qy, Tx, Tx)]\}$$

(iii) either (P, T) or (Q, T) are compatible pair

and

(iv) T is continuous

Further, if

(v) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the three selfmaps such that the sequence $Px_0, Qx_1, Px_2, Qx_3, \ldots, Px_{2n}, Qx_{2n+1}, \ldots$ converge to some point $z \in X$,

then P, Q and T have a unique common fixed point $z \in X$.

Before we give the proof of theorem, we establish some lemmas.

2.1. 1 Lemma: Suppose P, Q and T are selfmaps of a D*- metric space (X, D*) satisfying the conditions (i), (ii), (iv) and (v) of theorem 2.1. Then for any associated sequence $\{x_n\}$ of x_0 relative to P, Q and T we have

(a) $\lim_{n\to\infty} \lambda(Tx_{2n}, x_{2n+1}) = D^*(z, Tz, Tz)$ if (P, T) is compatible

and

(b) $\lim_{n\to\infty} \lambda(x_{2n}, Tx_{2n+1}) = D^*(z, Tz, Tz)$ if (Q, T) is compatible

Proof: Since by (v), each of the sequences $\{Px_{2n}\}$ and $\{Qx_{2n+1}\}$ converge to $z \in X$ and since $Px_{2n} = Tx_{2n+1}$ and $Qx_{2n+1} = Tx_{2n+2}$ for $n \geq 0$, we have

(2. 1. 2) $Px_{2n}, Qx_{2n+1}, Tx_{2n}, Tx_{2n+1} \rightarrow z$ as $n \rightarrow \infty$.

Now since T is continuous, we have

(2.1.3) $TPx_{2n} \rightarrow Tz$, $T^2x_{2n} \rightarrow Tz$ as $n \rightarrow \infty$

(a) If the pair the pair (P, T) is compatible, we have

(2.1.4) $\lim_{n\to\infty} D*(PTx_{2n}, TPx_{2n}, TPx_{2n}) = 0$

since $Px_{2n}, Tx_{2n} \rightarrow z$ as $n \rightarrow \infty$ by (2. 1. 2).

Now, in view of (2. 1. 3) and (2. 1. 4), we get

(2.1.5) $PTx_{2n} \rightarrow Tz$ as $n \rightarrow \infty$.

Also, from (ii)′, we have

(2. 1. 6) $\lambda(Tx_{2n}, x_{2n+1}) = \max\{D^*(T^2x_{2n}, Tx_{2n+1}, Tx_{2n+1}), D^*(PTx_{2n}, T^2x_{2n}, T^2x_{2n}),$

$D^*(Qx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[D^*(PTx_{2n}, Tx_{2n+1}, Tx_{2n+1}) + D^*(Qx_{2n+1}, T^2x_{2n}, T^2x_{2n})]\}$

Letting n to ∞ in (2. 1. 6) and using the continuity of D*, (2. 1. 2), (2. 1. 3), (2. 1. 4) and (2. 1. 5) we get

$\lim_{n\to\infty} \lambda(Tx_{2n}, x_{2n+1}) = \max\{D^*(Tz, z, z), D^*(Tz, Tz, Tz), D^*(z, z, z),$

$\frac{1}{2}[D^*(Tz, z, z) + D^*(z, Tz, Tz)]\}$

$= D^*(z, Tz, Tz).$

(b) If the pair the pair (Q, T) is compatible, we have

(2. 1. 7) $\lim_{n\to\infty} D*(TQx_{2n+1}, QTx_{2n+1}, QTx_{2n+1}) = 0$

in view of (2. 1. 2). Also since T is continuous, we have again by (2. 1. 2),

(2. 1. 8) $T^2x_{2n+1} \rightarrow Tz$ and $TQx_{2n+1} \rightarrow Tz$ as $n \rightarrow \infty$.

Now, in view of (2. 1. 7) and (2. 1. 8), we get

(2. 1. 9) $QTx_{2n+1} \rightarrow Tz$ as $n \rightarrow \infty$.

Now, from (ii)′, we have

(2. 1. 10) $\lambda(x_{2n}, Tx_{2n+1}) = \max\{D^*(Tx_{2n}, T^2x_{2n+1}, T^2x_{2n+1}), D^*(Px_{2n}, Tx_{2n}, Tx_{2n}),$

$D^*(QTx_{2n+1}, T^2x_{2n+1}, T^2x_{2n+1}), \frac{1}{2}[D^*(Px_{2n}, T^2x_{2n+1}, T^2x_{2n+1}) + D^*(QTx_{2n+1}, Tx_{2n}, Tx_{2n})]\}$

Now letting n to ∞ in (2. 1. 10) and using the continuity of D*, (2. 1. 2), (2. 1. 8) and (2. 1. 9), we get $\lim_{n\to\infty} \lambda(x_{2n}, Tx_{2n+1}) = \max\{D^*(z, Tz, Tz), D^*(z, z, z), D^*(Tz, Tz, Tz), \frac{1}{2}[D^*(z, Tz, Tz) + D^*(z, Tz, Tz)]\}$

$= D^*(z, Tz, Tz)$. Hence the lemma.

2. 2 Prof of Theorem 2. 1: In this section we first prove the existence of a common fixed point in the two cases of the condition (iii) in Theorem 2. 1.

Case (I). First suppose that the pair (P, T) is compatible. Then from (ii), we have

(2. 2. 1) $D^*(PTx_{2n}, Qx_{2n+1}, Qx_{2n+1}) \leq \varnothing(\lambda(Tx_{2n}, x_{2n+1}))$

In which on letting n to ∞ using Lemma 2. 1. 1, and the continuity of D*, we get

(2. 2. 2) $D^*(Tz, z, z) \leq \varnothing(D^*(Tz, z, z))$

and this leads to a contradiction if $Tz \neq z$. Therefore $Tz = z$.

Again, from condition (ii), we have

(2. 2. 3) $D^*(Pz, Qx_{2n+1}, Qx_{2n+1}) \leq \varnothing(\lambda(z, x_{2n+1}))$. But

$\lambda(z, x_{2n+1}) = \max\{D^*(Tz, Tx_{2n+1}, Tx_{2n+1}), D^*(Pz, Tz, Tz),$

$D^*(Qx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[D^*(Pz, Tx_{2n+1}, Tx_{2n+1}) + D^*(Qx_{2n+1}, Tz, Tz)]\}$

which on letting n to ∞ and the use of continuity of D* imply

$\lim_{n\to\infty} \lambda(z, x_{2n+1}) = D^*(z, Pz, Pz)$,

Now letting n to ∞ in (2. 2. 3), we get by the continuity of D* that

(2. 2. 4) $D^*(Pz, z, z) \leq \varnothing(D^*(Pz, z, z))$

and this leads to a contradiction if $Pz \neq z$. Therefore $Pz = z$.

Now again, from condition (ii), we have

(2. 2. 5) $D^*(Px_{2n}, Qz, Qz) \leq \varnothing(\lambda(x_{2n}, z))$. But

$\lambda(x_{2n}, z) = \max\{D^*(Tx_{2n}, Tz, Tz), D^*(Px_{2n}, Tx_{2n}, Tx_{2n}),$

$D^*(Qz, Tz, Tz), \frac{1}{2}[D^*(Px_{2n}, Tz, Tz) + D^*(Qz, Tx_{2n}, Tx_{2n})]\}$

in which on letting n to ∞ and the continuity of D*, we get

$\lim_{n\to\infty} \lambda(x_{2n}, z) = D^*(z, Qz, Qz)$, since $Px_{2n} \to z$, $Tx_{2n} \to z$ as $n \to \infty$. Then (2. 2. 5) gives

(2. 2. 6) $D^*(z, Qz, Qz) \leq \emptyset\,(D^*(z, Qz, Qz))$

and this will give a contradiction if $Qz \neq z$. Therefore $Qz = z$.

Hence $z = Pz = Qz = Tz$, showing that z is a common fixed point of P, Q and T.

Case (ii): Suppose that the pair (Q, T) is compatible, then from (ii), we have

(2. 2. 7) $D^*(Px_{2n}, QTx_{2n+1}, QTx_{2n+1}) \leq \emptyset\,(\lambda(x_{2n}, Tx_{2n+1}))$

in which on letting n to ∞ using Lemma 2. 1. 1, (2. 1. 9), and the continuity of D*, we get

(2. 2. 8) $D^*(z, Tz, Tz) \leq \emptyset\,(D^*(z, Tz, Tz))$

and this will be a contradiction if $Tz \neq z$. Therefore $Tz = z$.

Again, from condition (ii), we have

(2. 2. 9) $D^*(Pz, Qx_{2n+1}, Qx_{2n+1}) \leq \emptyset\,(\lambda(z, x_{2n+1}))$. But

$$\lambda(z, x_{2n+1}) = \max\{D^*(Tz, Tx_{2n+1}, Tx_{2n+1}), D^*(Pz, Tz, Tz), D^*(Qx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \tfrac{1}{2}[D^*(Pz, Tx_{2n+1}, Tx_{2n+1}) + D^*(Qx_{2n+1}, Tz, Tz)]\}$$

so that $\lim_{n\to\infty} \lambda(z, x_{2n+1}) = D^*(z, Pz, Pz)$,

Therefore, from (2. 2. 9), we have

(2. 2. 10) $D^*(Pz, z, z) \leq \emptyset\,(D^*(Pz, z, z))$

and this leads to a contradiction if $Pz \neq z$. Therefore $Pz = z$.

Again, from condition (ii), we have

(2. 2. 11) $D^*(Px_{2n}, Qz, Qz) \leq \emptyset\,(\lambda(x_{2n}, z))$. But

$$\lambda(x_{2n}, z) = \max\{D^*(Tx_{2n}, Tz, Tz), D^*(Px_{2n}, Tx_{2n}, Tx_{2n}), D^*(Qz, Tz, Tz), \tfrac{1}{2}[D^*(Px_{2n}, Tz, Tz) + D^*(Qz, Tx_{2n}, Tx_{2n})]\}$$

in which on letting n to ∞ we get $\lim_{n\to\infty} \lambda(x_{2n}, z) = D^*(z, Qz, Qz)$,

since $Px_{2n} \to z$, $Tx_{2n} \to z$ as $n \to \infty$ and $Pz = z = Tz$,

Then (2. 2. 11) gives

(2. 2. 12) $D^*(z, Qz, Qz) \leq \emptyset\,(D^*(z, Qz, Qz))$

and this will give a contradiction if $Qz \neq z$. Therefore $Qz = z$.

Hence $z = Pz = Qz = Tz$, showing that z is a common fixed point of P, Q and T

Now, we prove the **uniqueness** of the common fixed point. If possible, let z' be another common fixed point of P, Q and T. Then from condition (ii), we have

(2. 2. 13) $D^*(z, z', z') = D^*(Pz, Qz', Qz') \leq \Phi(\lambda(z, z'))$. But

$$\lambda(z, z') = \max \{D^*(Tz, Tz', Tz'), D^*(Pz, Tz, Tz), D^*(Qz', Tz', Tz'), \frac{1}{2}[D^*(Pz, Tz', Tz') + D^*(Qz', Tz, Tz)]\}$$

$= D^*(z, z', z')$. Therefore (2. 2. 13) gives

(2. 2. 14) $D^*(z, z', z') \leq \Phi(D^*(z, z', z'))$ and this will be contradiction if $z \neq z'$. Therefore $z = z'$. Thus z is the unique common fixed point of P, Q and T.

Thus the Theorem 2. 1 is completely proved.

2.3 A Common Fixed Point Theorem for Three Selfmaps of a Complete D*- metric space:

Before we prove the main result of this section, we prove the following lemma:

2.3.1 Lemma: Let (X, D*) be a D*- metric space and P, Q and T be selfmaps of X such that

(i) $P(X) \cup Q(X) \subseteq T(X)$

(ii) $D^*(Px, Qy, Qy) \leq c.\ \lambda(x, y)$ for all $x, y \in X$

where $0 \leq c <$ and $\lambda(x, y)$ is as defined in (ii)′ of Theorem 2. 1

Further, if

(iii) (X, D*) is complete.

Then for any $x_0 \in X$ and for any of its associated sequence $\{x_n\}$ relative to the three selfmaps, the sequence $Px_0, Qx_1, Px_2, Qx_3, \ldots, Px_{2n}, Qx_{2n+1}, \ldots$ converges to some $z \in X$.

Proof: Suppose P, Q and T be selfmaps of a D*-metric space (X, D*) for which the conditions (i) and (ii) hold. Let $x_0 \in X$ and $\{x_n\}$ be an associated sequence of x_0 relative to three selfmaps. Then, since $Px_{2n} = Tx_{2n+1}$ and $Qx_{2n+1} = Tx_{2n+2}$ for $n \geq 0$. Note that

$$\lambda(x_{2n}, x_{2n+1}) = \max \{D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}), D^*(Px_{2n}, Tx_{2n}, Tx_{2n}), D^*(Qx_{2n}, Tx_{2n+1}, Tx_{2n+1}), \frac{1}{2}[D^*(Px_{2n}, Tx_{2n+1}, Tx_{2n+1}) + D^*(QTx_{2n+1}, Tx_{2n}, Tx_{2n})]\}$$

$= \max \{D^*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}), D^*(Tx_{2n+1}, Tx_{2n}, Tx_{2n}), D^*(Tx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}),$

$\frac{1}{2}$ [D*(Tx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}) + D*(Tx_{2n+2}, Tx_{2n}, Tx_{2n})]} =

max {D*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}), D*(Tx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}), $\frac{1}{2}$ D*(Tx_{2n+2}, Tx_{2n}, Tx_{2n})}

Since $\frac{1}{2}$ D*(Tx_{2n+2}, Tx_{2n}, Tx_{2n}) = $\frac{1}{2}$ [D*(Tx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}) + D*(Tx_{2n+1}, Tx_{2n}, Tx_{2n})]}

≤ max {D*(Tx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}), D*(Tx_{2n+1}, Tx_{2n}, Tx_{2n}),

$\lambda(x_{2n}, x_{2n+1})$ ≤ max { D*(Tx_{2n}, Tx_{2n+1},Tx_{2n+1}), D*(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2})}

Now by (ii)

D*(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2}) = D*(Px_{2n}, Qx_{2n+1}, qx_{2n+1}) ≤ c. $\lambda(x_{2n}, x_{2n+1})$

≤ c. max { D*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}), D*(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2})}.

Since $0 \leq c < 1$, it follows from that the

max { D*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}), D*(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2})} = D*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1})

Therefore D*(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2}) ≤ c. D*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}) …….. (A)

Similarly, we can prove

D*(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}) ≤ c. D*(Tx_{2n}, Tx_{2n-1}, Tx_{2n-1}) …….. (B)

From (A) and (B), we get

D*(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2}) ≤ c2 D*(Tx_{2n}, Tx_{2n-1}, Tx_{2n-1})

≤ c^4 D*(Tx_{2n-1}, Tx_{2n-3}, Tx_{2n-3})

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≤ c^{2n} D*(Tx_2,Tx_0,Tx_0) → 0

Since $c^{2n} \to 0$ as $n \to \infty$ (because $c < 1$), the sequence {Tx_n} and hence Px_0, Qx_1, Px_2, Qx_3,….., Px_{2n}, Qx_{2n+1}….. is a Cauchy sequence in the complete space (X, D*) and therefore converges to a point say z ϵ X, proving lemma.

2.3.2 Remark: The converse of lemma is not true. That is, suppose P, Q and T are selfmaps of a D*-metric space (X, D*) satisfying condition (i) and (ii) of Lemma 2.3.1. Even, if for each x_0 ϵ X and for each associated sequence {x_n} of x_0 relative to P, Q and T, the sequence Px_0, Qx_1, Px_2, Qx_3, ….., Px_{2n}, Qx_{2n+1}, ….. converges in X, then (X, D*) need not complete.

2. 3. 3 Theorem: Suppose (X, D*) is a D*-metric space satisfying conditions (i) to (iv) of Theorem 2. 1. Further, if (v)′ (X, D*) is complete

then P, Q and T have a unique common fixed point zϵ X.

Proof: In view of Lemma 2.3.1 the condition (v) of Theorem 2.1 holds as view of (v)'.

Hence the Theorem follows from Theorem 2.1.

2.3.4 Corollary ([9]): Let P, Q and T be selfmaps of a metric space (X, d) such that

(i) $P(X) \cup Q(X) \subseteq T(X)$

(ii) $d(Px, Qy) \leq c\, \lambda_0 (x, y))$ for all x, y ϵ X,

where

(ii)' $\lambda_0(x, y) = \max \{d(Tx, Ty), d(Px, Tx), d(Qy, Ty), \frac{1}{2}[d(Px, Ty) + d(Qy, Sx)]\}$ and $0 \leq c < 1$

(iii) T is continuous, and

(iv) PT = TP and QT = TQ

Further, if

(v) X is complete

Then P, Q and T have a unique common fixed point in zϵ X.

Proof: Given (X, d) is a metric space satisfying condition (i) to (v) of the corollary. If $D_1^*(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)$, then (X, D_1^*) is a D*-metric space and $D_1^*(x, y, x) = d(x, y)$. Therefore condition (ii) can be written as $D_1^*(Px, Qy, Qy) \leq c.\ \lambda(x, y)$ for all x, y ϵ X where $\lambda(x, y) = \max \{D_1^*(Tx, Ty, Ty), D_1^*(Px, Tx, Tx), D_1^*(Qy, Jy, Jy),$

$$\frac{1}{2}[D_1^*(Px, Ty, Ty) + D_1^*(Qy, Tx, Tx)]\}$$

which is the same as condition (ii) of Theorem 2.3.3. Also since (X, d) is complete, we have (X, D_1^*) is complete by Corollary1.13.

Now, P, Q and T are selfmaps on (X, D_1^*) satisfying conditions of Theorem 2.3.3 and hence the corollary follows.

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